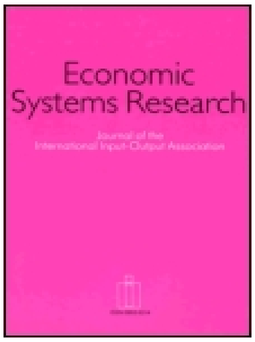

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MATRIX BALANCING UNDER CONFLICTING INFORMATION

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We have developed a generalised iterative scaling method (KRAS) that is able to balance and reconcile input–output tables and SAMs under conflicting external information and inconsistent constraints. Like earlier RAS variants, KRAS can: (a) handle constraints on arbitrarily sized and shaped subsets of matrix elements; (b) include reliability of the initial estimate and the external constraints; and (c) deal with negative values, and preserve the sign of matrix elements. Applying KRAS in four case studies, we find that, as with constrained optimisation, KRAS is able to find a compromise solution between inconsistent constraints. This feature does not exist in conventional RAS variants such as GRAS. KRAS can constitute a major advance for the practice of balancing input–output tables and Social Accounting Matrices, in that it removes the necessity of manually tracing inconsistencies in external information. This quality does not come at the expense of substantial programming and computational requirements (of conventional constrained optimisation techniques).

Keywords: Matrix balancing; Inconsistent constraints; Constrained optimisation; RAS

1 INTRODUCTION

A common problem in compiling and updating Social Accounting Matrices (SAM) or input–output tables is that of incomplete data. Missing matrix elements may be due to a variety of reasons, such as costly and therefore incomplete industry surveys, or the suppression of confidential information. External data points can be used to formulate a system of equations that constrain the unknown matrix elements. However, unknowns usually outnumber external constraints, resulting in the system being underdetermined, that is exhibiting too many degrees of freedom to be solved analytically. The two most prominent numerical approaches for reconciling such an underdetermined system are probably the *RAS bi-proportional scaling* method, and mathematical programming methods summarised here under the term *constrained optimisation*.

During the past 40 years, both approaches have successfully tackled a number of challenges, leading to a number of useful features (Lahr and de Mesnard, 2004; Huang et al., 2008).[1] Ideally, the technique should:

a) incorporate constraints on arbitrarily sized and shaped subsets of matrix elements, instead of only fixing row and column sums;

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[1] Lahr and de Mesnard (2004) provide a recent overview of extensions to the classic RAS technique. Huang et al. (2008) explore different objective functions for input-output matrix updating.

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b) allow considering the reliability of the initial estimate;
c) allow considering the reliability of external constraints;
d) be able to handle negative values and to preserve the sign of matrix elements if required;
e) be able to handle conflicting external data.

While all criteria have been addressed by constrained optimisation methods, there is currently no RAS-type iterative scaling technique that satisfies criterion (e) (Canning and Wang, 2005; Cole, 1992).[2] The inability of RAS to deal with conflicting external data represents a considerable drawback for practice, because for most statistical agencies such data are often rather the norm than the exception.

The most simple case of conflicting data is probably a situation in which two data sources are located that prescribe two different values for the same matrix entry, resulting in inconsistent constraints. When faced with such constraints, existing RAS variants adjust the respective matrix element alternatively to either of the conflicting values, and thus enter into oscillations without ever converging to a satisfactory solution.

More generally, sets of external data can be conflicting indirectly amongst each other. Cole mentions convergence problems, and gives a simple example as a matrix (Cole, 1992)

$$\begin{pmatrix} a & b \\ c & 1 \end{pmatrix}$$

with $a, b, c \geq 0$, and with inconsistent row and column totals $(1, 3)'$ and $(1, 3)$. Möhr et al. (1987) provide an example where the RAS-infeasibility is brought about by the existence of unadjustable zero values in the matrix to be balanced.[3] In practice, indirect conflict might present itself for example when, on one hand, data on final demand and gross output of wheat suggest a certain intermediate demand of wheat, however on the other hand this intermediate demand is too large to be absorbed by the flour milling sector. Further examples involving conflicting external information are GDP measures, and multi-national and regional input–output systems (Barker et al., 1984; Smith et al., 1998).[4] Möhr et al. call such problems "RAS-infeasible". In practice, such inconsistencies are often traced and adjusted manually by statisticians (Dalgaard and Gnysting, 2004).[5]

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[2] Recently, Canning and Wang (2005) note that "another important advantage of the mathematical programming model over scaling methods is in its flexibility. Additional constraints can be easily imposed, such as [...] incorporating an associate term in the objective function to penalize solution deviations from the initial row and column total estimates when they are not known with certainty."

[3] Möhr et al. (1987) make this problem feasible by adding an "augmentation" which ensures that matrix elements are non-zero where constraints require it. While this is a solution to these authors' problem, it does not address constraint conflicts in general.

[4] Barker et al. (1984, p. 475) write: "... we observed that the income, expenditure, production and financial estimates of data are typically inconsistent. The presence of such accounting inconsistencies emphasises the unreliable nature of economic data." See also Smith et al. (1998).

[5] Barker et al. (1984, p. 475) remark that "... trading off the relative degrees of uncertainty of the various data items in the system in order to adjust the prior data to fit the accounting identities [...] is essentially what national income accountants do during the last stages of compiling the accounts when faced with major discrepancies between data from different sources". Dalgaard and Gysting (2004, p. 170) from Statistik Denmark report that many analysts responsible for compiling input–output tables favour manual adjustment, because "based on the experience that many errors in primary statistics are spotted in the course of a balancing process that is predominantly manual,

This work has two aims: (1) we undertake a comparative review of the development of both RAS and constrained optimisation techniques; (2) we present a new generalised iterative scaling method that is able to handle all five criteria above at once, and especially conflicting external data and inconsistent constraints. We achieve this capability by introducing standard error estimates for external data. We build on previous RAS variants that satisfy the remaining criteria, and thus arrive at a RAS-type method that matches the capabilities of constrained optimisation. We will refer to our method as KRAS (Konfliktfreies RAS).

2 LITERATURE REVIEW

The following review is organised according to the criteria list in the introduction: Section 2.1 deals with criterion (a), Section 2.2 with (b) and (c), Section 2.3 with (d), and Section 2.4 with (e).

The RAS method – in its basic form – bi-proportionally scales a matrix $\mathbf{A}_0$ of unbalanced preliminary estimates of an unknown real matrix **A**, using **A**'s known row and column sums. The balancing process is usually aborted when the discrepancy between the row and column sums of $\mathbf{A}_0$ and **A** is less than a previously fixed threshold. Bacharach has analysed the bi-proportional constrained matrix problem in great detail, in particular with regard to the economic meaning of bi-proportional change (Bacharach, 1970; Leontief, 1941; Stone and Brown, 1962; Miernyk, 1976; Giarratani, 1975),[6] the existence and uniqueness of the iterative RAS solution, its properties of minimisation of a distance metric (Robinson et al., 2001),[7] and uncertainty associated with errors in row and column sum data and with the assumption of biproportionality. The origins of the method go back several decades (Kruithof, 1937; Lamond and Stewart, 1981; Deming and Stephan, 1940; Bregman, 1967).[8] Stone and Brown (1962), Bacharach (1970), and Polenske (1997) provide a historical background.

2.1 Constraints on Arbitrarily Sized and Shaped Subsets of Matrix Elements

The "modified RAS" (MRAS) approach was developed for cases when some of the matrix elements of **A** are known in addition to its row and column sums, for example from an

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compilers are typically convinced that a (mainly) manual balancing process yields results of higher quality than those emanating from a purely automatic balancing of the accounts. From that point of view, the resources involved in manual balancing are justified as a very efficient consistency check on the accounts."

[6] When applied to the forecasting of monetary input–output matrices, bi-proportional changes have been interpreted as productivity, substitution or fabrication effects (Leontief, 1941; Stone and Brown, 1962) affecting industries over time. Miernyk's (1976) view however is that the RAS method "substitutes computational tractability for economic logic", and that the production interpretation loses its meaning when the entire input–output table is balanced, and not only inter-industry transactions (see also Giarratani, 1975).

[7] The RAS, Linear Programming and minimum information gain algorithms yield a balanced matrix estimate that is – in terms of some measure of multidimensional "distance" – closest to the unbalanced preliminary estimate. When applied to temporal forecasting, this property is explained as a conservative hypothesis of attributing inertia to inter-industrial relations (Bacharach, 1970, p. 26). While the classic RAS method is aimed at maintaining the value structure of the balanced matrix, the closely related cross-entropy methods (Robinson et al., 2001) are aimed at maintaining the coefficient structure.

[8] See Kruithof (1937) as cited in Lamond and Stewart (1981); Deming and Stephan (1940); attributed to 1930s Leningrad architect Sheleikhovskii by Bregman (1967a).

industry survey (Paelinck and Waelbroeck, 1963). MRAS "nets" the preliminary estimate $\mathbf{A}_0$, that is the known elements are subtracted, and $\mathbf{A}_0$ contains 0 at the corresponding positions. The net $\mathbf{A}_0$ is then subjected to the standard RAS procedure, and the known elements are added back on after balancing.

In addition to certain elements of $\mathbf{A}$, some aggregates of elements of $\mathbf{A}$ may be known. For example, a published table $\mathbf{A}^{\mathrm{G}}$ of national aggregates may constitute partial information when constructing a multi-regional input–output system, or a more disaggregated national table. Accordingly, Oosterhaven et al. (1986) add a "national cell constraint" to the standard row and column sum constraints. Similarly, Jackson and Comer (1993) use partition coefficients for groups of cells of a disaggregated base year matrix to disaggregate cells in an updated but aggregated matrix. Batten and Martellato (1985) discuss further constraints structures, involving intermediate and final demand data. Gilchrist and St Louis (1999, 2004) propose a three-stage "TRAS" for the case when aggregation rules exist under which the partial aggregated information $\mathbf{A}^{\mathrm{G}}$ can be constructed from its disaggregated form $\mathbf{A}$. Cole (1992) describes the general TRAS type that accepts constrained subsets of any size or shape. Gilchrist and St Louis (1999), as well as Lenzen et al. (2006) demonstrate that the inclusion of partial aggregated information into the RAS procedure leads to superior outcomes.

2.2 Reliability of the Initial Estimate and External Information

Another variant of the MRAS method takes into account the uncertainty of the preliminary estimates, and contains the occurrence of perfectly known elements as a special case (see Lecomber, 1975a, with case studies in Allen, 1974, and Allen and Lecomber, 1975). This is accomplished by introducing a matrix $\mathbf{E}$ containing "reliability information" about the elements in $\mathbf{A}_0$. $\mathbf{E}$ instead of $\mathbf{A}_0$ is then balanced in order to take up the difference between the preliminary and true totals:

$$\mathbf{A}^* = (\mathbf{A}_0 - \mathbf{E}) + \hat{\mathbf{r}}\mathbf{E}\hat{\mathbf{s}} \tag{1}$$

$\mathbf{A}^*$ is the balanced estimate, and $\hat{\mathbf{r}}$ and $\hat{\mathbf{s}}$ are diagonal scaling matrices, as in the conventional RAS algorithm. Where $e_{ij} = 0$, a_{ij} remains unchanged during balancing. Lecomber also investigates the influence of errors in the 'true' totals (Lecomber 1975a; 1975b).

A shortcoming of Lecomber's approach is that the elements of $\mathbf{E}$ cannot be interpreted as standard deviations. If we follow Lecomber in maintaining $0 \leq e_{ij} \leq a_{0ij}$, and consider that RAS preserves the positive signs in $\mathbf{E}$, then $a^*_{ij} \geq a_{0ij} - e_{ij}, \forall ij$. In other words if e_{ij} were the standard deviations of a_{0ij}, then the balanced estimate $\mathbf{A}^*$ could never go below the initial estimate $\mathbf{A}_0$ for more than one standard deviation. An upper limit for $\mathbf{A}^*$ does not exist however. Thus, as Lecomber points out, the elements of $\mathbf{E}$ must be sufficiently large to ensure the controlling vectors are non-negative – but there is no method to ensure this, whilst still interpreting the elements of $\mathbf{E}$ as standard errors. Thus, considering that conflicting external information may well diverge by more than one standard deviation, it follows that MRAS will not reach a solution under sufficiently inconsistent constraints, unless more (unspecified) information on errors is obtained.

Lahr takes into account the uncertainties of external constraints in treating the tolerances of the RAS termination criteria as functions of the varying reliabilities of row and column sums (Lahr, 2001). Dalgaard and Gysting (2004) incorporate information about the reliability of external constraints (again row and column totals) into the balancing process as "confidence factors" $\boldsymbol{\lambda}$, and successively adjust the target totals $\mathbf{u}^{(n)}$ of the nth iteration as a weighted sum $u_j^{(n)} = \lambda_j^{(n-1)} u_j^{(0)} + \left(1 - \lambda_j^{(n-1)}\right) u_j^{(n-1)}$ of the initial unbalanced totals $u_j^{(0)}$ and the totals $u_j^{(n-1)}$ of the previous iteration. With subsequent iterations, the confidence factors $0 \leq \lambda_j^{(h)} \leq 1$ become smaller and smaller, thus gradually converging away from the unbalanced initial totals $\mathbf{u}^{(0)}$, towards the balanced totals $\mathbf{u}^{(\infty)}$. The innovation is that totals with high confidence ($\lambda_j \leq 1$) get adjusted away from the initial totals much slower than those totals with low confidence ($\lambda_j \geq 0$).

While both approaches consider the varying reliability of totals, they cannot deal with inconsistent totals. In applying conventional RAS scaling factors, Lahr's algorithm would always end up balancing matrix elements to satisfy only one of a number of conflicting external constraints. Similarly, for large enough n, Dalgaard and Gysting's (2004) algorithm would oscillate around those inconsistent totals $u_j^{(n-1)}$ with non-zero confidence.[9]

2.3 Negative Elements

Junius and Oosterhaven (2003) derive a generalised RAS ("GRAS") algorithm that can balance negative elements, by splitting the matrix **A** into positive and negative parts **P** and **N**, and balancing **A**=**P** − **N** according to

$$\begin{aligned} \left(\hat{\mathbf{r}}\mathbf{P}\hat{\mathbf{s}} - \hat{\mathbf{r}}^{-1}\mathbf{N}\hat{\mathbf{s}}^{-1}\right)\mathbf{i} &= \mathbf{u}^* \\ \mathbf{i}\left(\hat{\mathbf{r}}\mathbf{P}\hat{\mathbf{s}} - \hat{\mathbf{r}}^{-1}\mathbf{N}\hat{\mathbf{s}}^{-1}\right) &= \mathbf{v}^* \end{aligned} \tag{2}$$

where $\mathbf{i}$ is the summation vector, and $\mathbf{u}^* = e\mathbf{u}$ and $\mathbf{v}^* = e\mathbf{v}$ (Lenzen et al., 2007; Junius and Oosterhaven, 2003).[10]

2.4 Conflicting Data–Constrained Optimisation

Neither MRAS, TRAS, nor GRAS deal satisfactorily with uncertainty and data conflict. Such problems are so far only solved by constrained optimisation methods. Bacharach (1970) has already shown that the conventional RAS technique is equivalent to the constrained minimisation of an information gain function $f = \Sigma_{ij}[a_{ij}\ln(a_{ij}/ea_{0ij})]$. Naturally, this circumstance led to the parallel developments of both RAS and constrained optimisation techniques for the purpose of balancing input–output tables or SAMs. It is interesting to see that researchers working on either technique have faced almost the same challenges.

The basic structure of a constrained optimisation problem applied to SAMs is

$$\text{Minimise } f(\mathbf{A}, \mathbf{A}_0), \text{ subject to } \Sigma_i a_{ij} = x_j \text{ and } \Sigma_j a_{ij} = x_i \tag{3}$$

[9] Dalgaard and Gysting (2004) do describe balancing matrices with "unequal net row and column sum" and "macro differences between supply and use". However, rather than inconsistencies in external information, this means correct differences in the sum over supply by *industry* and use by *product*, which naturally occur in asymmetric commodity-by-industry supply and use tables.

[10] Using the trivial case of starting with an initial estimate **A** that already satisfies all prescribed row and column totals, Lenzen et al. (2007) construct a case where Junius and Oosterhaven's (2003) GRAS balancing algorithm leads to a solution **X** that is inferior to the initial estimate in terms of their target function. They show that the factor e has to be taken out of the Junius and Oosterhaven formulation in order to correct the problem.

TABLE 1. Recent extensions to RAS and optimisation techniques for balancing SAMs and input–output tables.

Criterion	RAS-type technique	Constrained optimisation
(a)	Gilchrist and St Louis (1999)	Morrison and Thumann (1980)
(b)	Lecomber (1975a, 1975b)	Stone et al. (1942) Byron (1978)
(c)	Lecomber (1975a; 1975b); Lahr (2001), Dalgaard and Gysting (2004)	van der Ploeg (1982)
(d)	Junius and Oosterhaven (2003)	Harrigan and Buchanan (1970)
(e)	This work	van der Ploeg (1982)

where f is the objective function, and x_i and x_j are row and column totals. Morrison and Thumann (1980) minimise a weighted sum of squares of deviations $f = \Sigma_{ij} (a_{ij} - a_{0ij})^2 / w_{ij}$, where the w_{ij} are the weights. They also explicitly describe the incorporation of external information referring to general subsets of matrix elements, into a Lagrange multiplier approach.

Byron (1978) incorporates variances for the initial estimate into a quadratic Lagrange function, and uses the first-order conditions to solve for the Lagrange multipliers and the balanced SAM. Van der Ploeg (1982; 1984) elegantly extends Byron's formulation by (a) adding disturbances to the external constraint information; and (b) combining the SAM vector with these disturbances, and solve the for both the SAM and the disturbances simultaneously. Since the solution for the Lagrange multipliers involves matrix inversion, computing times are strongly influenced by the sizes of the SAM and constraint system. Both Byron and van der Ploeg go to great lengths in exploiting the sparse structure of the coefficients matrix, and in devising efficient algorithms in order to be able to solve large SAMs. In effect, it is the introduction of the variance and disturbance terms that enables handling conflicting external data (van der Ploeg calls it "constraint violation"), because the solutions for the constraint values are allowed to deviate from its prescribed value.

Lecomber (1975a), Morrison and Thumann (1980), and Harrigan and Buchanan (1984) note that the conventional Langrange multiplier procedure (Equation 3) does not guarantee non-negative solutions. This is undesirable because negative matrix entries can present problems in input–output analysis (ten Raa and Van der Ploeg, 1989). With the requirement of non-negativity, the constrained optimisation problem essentially becomes a *bounded* constrained optimisation. In general, one asks that the unknown SAM elements are within lower and upper bounds $l_i \leq a_i \leq u_i$.[11]

The mixing of equality and inequality conditions requires quadratic programming methods, which renders the solution of the optimisation problem considerably more complicated, as the expositions of Harrigan and Buchanan (1984), Zenios et al. (1989), and Nagurney and Robinson (1992) may testify. This circumstance provides the motivation for searching for a RAS variant that is able to deal with conflicting external data, (see Table 1).

[11] Tarancon and Del Rio (2005) present an interesting variant of the bounded optimisation problem, by deriving lower and upper bounds from criteria for the stable structural evolution of input–output coefficients, and introducing supplementary variables to take up the slack between the bounds and the matrix entries. If the model turns out to be inconsistent because some constraints cannot be met within those bounds, then the analyst manually chooses certain constraints to be relaxed, until no variable exceeds the bounds.

3. KRAS – AN EXTENSION OF GRAS

Compared with constrained optimisation techniques, RAS has enjoyed higher popularity, which is probably due to ease of programming. However, as Tarancon and Del Rio (2005, p.2) explicitly state, at present "… the RAS process cannot be developed with interval estimates of the margins. Hence, point estimates are used, which may carry an implicit error." Considering that the use of RAS in statistical agencies requires the manual and therefore often tedious removal of inconsistencies in the constraint system, it would be desirable to have a RAS technique that deals with such common occurrences in a systematic and automated way. The description of such a RAS variant is the topic of this section. We will base our derivation strongly on the GRAS method of Junius and Oosterhaven (2003). Our exposition proceeds in three steps. First, we generalise of GRAS to constraints on arbitrary subsets of matrix elements. Second, we relax the restriction of GRAS for constraint coefficients to be either 1 or −1, and allow any real number. Third, we incorporate reliability and conflict of external data. While we present these extensions in a succession for the sake of clarity, each of them can be applied separately to GRAS. For example, it is possible to modify the original GRAS method only by incorporating the modifications in Section 0 in order to deal with reliability and conflict of external data.

Bacharach (1970, pp. 79–86) shows that the simple biproportional RAS algorithm can be derived from minimizing a minimum-information function

$$f(\mathbf{A}, \mathbf{A}_0) = \sum_{i,j} a_{ij} \ln \frac{a_{ij}}{e\, a_{0ij}} \tag{4}$$

subject to constraints **u** and **v** on known row and column totals

$$\sum_j a_{ij} = u_i \quad \text{and} \quad \sum_i a_{ij} = v_j \tag{5}$$

where e is the basis of the natural logarithm. The GRAS method is derived in the same way. However, the preliminary estimate $\mathbf{A}_0$ (which becomes the solution $\mathbf{A}^{(0)}$ at step zero), is split into positive and negative parts according to $\mathbf{A}^{(0)} = \mathbf{P}^{(0)} - \mathbf{N}^{(0)}$, and alternately row- and column-scaled using diagonal scalar matrices $\hat{\mathbf{r}}$ and $\hat{\mathbf{s}}$, so that after the nth round of balancing, $\mathbf{A}^{(n)} = \hat{\mathbf{r}}^{(n-1)}\mathbf{P}^{(n-1)}\hat{\mathbf{s}}^{(n-1)} - \left[\hat{\mathbf{r}}^{(n-1)}\right]^{-1}\mathbf{N}^{(n-1)}\left[\hat{\mathbf{s}}^{(n-1)}\right]^{-1}$. $\mathbf{A}^{(n)}$ is then subjected to the next scaling operation. Junius and Oosterhaven's GRAS derivation arrives at a second-order polynomial that defines scalars

$$\begin{aligned} r_i^{(n)} &= \frac{u_i + \sqrt{u_i^2 + 4\sum_j p_{ij}^{(n)} \sum_j n_{ij}^{(n)}}}{2\sum_j p_{ij}^{(n)}}, \quad \text{with} \\ p_{ij}^{(n)} &= p_{ij}^{(n-1)} s_j^{(n-1)} \\ n_{ij}^{(n)} &= n_{ij}^{(n-1)} \left[s_j^{(n-1)}\right]^{-1} \qquad s_j^{(n-1)} = \frac{v_j + \sqrt{v_j^2 + 4\sum_i p_{ij}^{(n-1)} \sum_i n_{ij}^{(n-1)}}}{2\sum_i p_{ij}^{(n-1)}} \end{aligned} \tag{6}$$

The algorithm converges if $\left\| \left(\hat{\mathbf{r}}\mathbf{P}\hat{\mathbf{s}} - \hat{\mathbf{r}}^{-1}\mathbf{N}\hat{\mathbf{s}}^{-1} \right)\mathbf{i} - \mathbf{u} \right\| < \delta \|\mathbf{u}\|$ and $\left\| \mathbf{i}\left(\hat{\mathbf{r}}\mathbf{P}\hat{\mathbf{s}} - \hat{\mathbf{r}}^{-1}\mathbf{N}\hat{\mathbf{s}}^{-1} \right) - \mathbf{v} \right\| < \delta \|\mathbf{v}\|$ for a sufficiently small δ.

3.1 Incorporating Constraints on Arbitrary Subsets of Matrix Elements

Consider now a generalised formulation of constraints as in $\mathbf{Ga} = \mathbf{c}$, where $\mathbf{a}$ is the vectorisation of $\mathbf{A}$ above. Such a formulation includes constrained row and column sums, constraint single elements, constrained subsets, and negative elements as special cases. Constraints can include any number of elements, which may be fully, partly or non-adjacent.[12] Constraints may also exclude some of the row and column totals (compare Thissen and Löfgren, 1998). The Junius and Oosterhaven minimisation problem then becomes

$$\text{Minimise } f(\mathbf{a}, \mathbf{a}_0) = \sum_{i,j} \left| a_{ij} \right| \ln \frac{a_{ij}}{e\, a_{0ij}} \text{ subject to } \mathbf{Ga} = \mathbf{c} \tag{7}$$

Let there be n_C constraints. Equation 6 can then be generalised to

$$\begin{aligned} r^{(n)} &= \frac{c_i + \sqrt{c_i^2 + 4 \sum_{j, a_j^{(n-1)} g_{ij} > 0} g_{ij} a_j^{(n-1)} \sum_{j, a_j^{(n-1)} g_{ij} < 0} g_{ij} a_j^{(n-1)}}}{2 \sum_{j, a_j^{(n-1)} g_{ij} > 0} g_{ij} a_j^{(n-1)}} \quad \text{and} \\ a_j^{(n)} &= a_j^{(n-1)} \left[r^{(n)} \right]^{\mathrm{Sgn}\left(a_j^{(n-1)} g_{ij} \right)}, \quad \text{with} \quad i = n \bmod n_C \end{aligned} \tag{8}$$

In Equation 8, the negative elements in Equation 6 have been replaced with negative coefficients on positive elements, but otherwise the formulation is exactly the same. There is only one scalar r_i for each constraint i, and these scalars are applied consecutively for all $i = 1, \ldots, n_C$.[13] The r_i and a_j are calculated alternately. The GRAS feature of scaling negative elements by the inverse of the positive scalar is evident in the exponent $\mathrm{Sgn}\left(a_j^{(n-1)} g_{ij} \right)$ in Equation 8. The algorithm converges if

$$\|\mathbf{Ga} - \mathbf{c}\| < \delta \|\mathbf{c}\| \tag{9}$$

for a sufficiently small δ.

3.2 Incorporating Constraints with Non-unity Coefficients

Consider now non-unity constraint coefficients $g_{ij} \in \mathbb{R}$.[14] For such constraints the Junius and Oosterhaven GRAS will not work anymore. Allowing the a_i to be both positive and

[12] Single-element constraints need not be part of the scaling procedure, but could be "netted out" using the "modified RAS" method.

[13] Here, the modulus function $a \bmod b$ refers to the remainder of the division of a by b.

[14] Non-unity constraints can appear when there is knowledge of relative as opposed to absolute values for some matrix elements. For example, in the construction of multi-regional input–output systems we may use information regarding the fraction of value added or gross output allocated to a given region.

negative (as in GRAS), we write the Lagrangean as

$$\boldsymbol{L}(\mathbf{a}, \mathbf{a}_0, \boldsymbol{\lambda}) = \sum_{i;a_i \geq 0} a_i \ln \frac{a_i}{e\, a_{0i}} - \sum_{i;a_i<0} a_i \ln \frac{a_i}{e\, a_{0i}} + \sum_i \lambda_i \left[\sum_j g_{ij} a_j - c_i \right] \tag{10}$$

The first-order minimum conditions are

$$0 = \frac{\partial \boldsymbol{L}(\mathbf{a}, \mathbf{a}_0, \boldsymbol{\lambda})}{\partial a_k} = \begin{cases} \ln \frac{a_k}{e\, a_{0k}} + 1 + \sum_i \lambda_i g_{ik} = \ln \frac{a_k}{a_{0k}} + \sum_i \lambda_i g_{ik} & \text{if } a_k \geq 0 \\ -\ln \frac{a_k}{e\, a_{0k}} - 1 + \sum_i \lambda_i g_{ik} = -\ln \frac{a_k}{a_{0k}} + \sum_i \lambda_i g_{ik} & \text{if } a_k < 0 \end{cases} \tag{11}$$

The solution for **a** is hence of the form

$$a_k = \begin{cases} a_{0k} e^{-\sum_i \lambda_i g_{ik}} = a_{0k} \prod_i e^{-\lambda_i g_{ik}} & \text{if } a_k \geq 0 \\ a_{0k} e^{\sum_i \lambda_i g_{ik}} = a_{0k} \prod_i e^{\lambda_i g_{ik}} & \text{if } a_k < 0 \end{cases} \tag{12}$$

This form reflects the fact that negative and positive matrix elements are proportionately balanced with respectively inverse scaling terms $\Pi_i e^{-\lambda_i g_{ik}}$ and $\Pi_i e^{\lambda_i g_{ik}}$. Inserting Equation (11) into the condition $\mathbf{Ga} = \mathbf{c}$ yields

$$\sum_{j;a_{0j} \geq 0} g_{ij} a_{0j} \prod_k e^{-\lambda_k g_{kj}} + \sum_{j;a_{0j}<0} g_{ij} a_{0j} \prod_k e^{\lambda_k g_{kj}} - c_i = 0 \tag{13}$$

The multi-proportional problem in Equation 13 can be solved iteratively using Bregman's balancing method (Bregman, 1967a; Elfving, 1980; Lamond and Stewart, 1981; Erlander, 1981).[15] Starting with the initial solution $\boldsymbol{a}^{(0)} = \boldsymbol{a}_0$, and with chosen $\{\lambda_k^{(0)}\}_{k=1,\ldots,n_C}$, only one of the n_C Lagrange multipliers λ_i is adjusted at any one time, by first solving constraint 1

$$\begin{aligned} 0a_j^{(0)} &\geq 0\, g_{ij} a_j^{(1)} e^{-\lambda_i^{(1)} g_{ij}} \prod_{k \neq i} e^{-\lambda_k^{(1)} g_{kj}} + \sum_{j;a_j^{(n-1)}<0} g_{ij} a_j^{(0)} e^{\lambda_k^{(1)} g_{ij}} \prod_{k \neq i} e^{\lambda_k^{(0)} g_{kj}} - c_i a_j^{(0)} \\ &\geq 0 g_{ij} a_j^{(0)} e^{-\left(\lambda_i^{(1)} - \lambda_i^{(0)}\right) g_{ij}} \prod_k e^{-\lambda_k^{(0)} g_{kj}} + \sum_{j;a_j^{(0)}<0} g_{ij} a_j^{(0)} e^{\left(\lambda_k^{(1)} - \lambda_i^{(0)}\right) g_{ij}} \prod_k e^{\lambda_k^{(0)} g_{kj}} - c_i \end{aligned} \tag{14}$$

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[15] See Bregman (1967b), Elfving (1980) and Lamond and Stewart (1981). Elfving (1980) distinguishes "Gauss-Seidel type schemes", where only one Lagrange multiplier is adjusted in every step, and "Jacobi type methods" where the non-linear system in Equation 12 is solved simultaneously for all Lagrange multipliers. The latter methods require the Jacobian of Equation 12 to have full rank, i.e. all constraint equations have to be linearly independent (compare Eriksson, 1980; Erlander, 1981).

then updating

$$a_j^{(1)} = \begin{cases} a_j^{(0)} e^{-\left(\lambda_i^{(1)} - \lambda_i^{(0)}\right) g_{ij}} \prod_k e^{-\lambda_k^{(0)} g_{kj}} \text{ for } a_j^{(0)} \geq 0 \\ a_j^{(0)} e^{\left(\lambda_i^{(1)} - \lambda_i^{(0)}\right) g_{ij}} \prod_k e^{\lambda_k^{(0)} g_{kj}} \text{ for } a_j^{(0)} < 0 \end{cases} \tag{15}$$

with $\lambda^{(1)}$ given by Equation 14. This is followed by solving constraints $i\ \forall i > 1$

$$\sum_{j;a_j^{(n-1)} \geq 0} g_{ij} a_j^{(n-1)} e^{-\left(\lambda_i^{(n)} - \lambda_i^{(n-1)}\right) g_{ij}} + \sum_{j;a_j^{(n-1)} < 0} g_{ij} a_j^{(n-1)} e^{\left(\lambda_k^{(n)} - \lambda_i^{(n-1)}\right) g_{ij}} - c_i = 0 \tag{16}$$

for$\lambda_i^{(n)}$, iteratively, with known $\lambda_k^{(n-1)}$ $\forall k$, and with i=n mod n_C. The new solution of each step (n) is computed via (compare Equations 2.5 and 2.6 in Elfving, 1980)

$$a_j^{(n)} = \begin{cases} a_j^{(n-1)} e^{-\left(\lambda_i^{(n)} - \lambda_i^{(n-1)}\right) g_{ij}} \text{ for } a_j^{(n-1)} \geq 0 \\ a_j^{(n-1)} e^{\left(\lambda_i^{(n)} - \lambda_i^{(n-1)}\right) g_{ij}} \text{ for } a_j^{(n-1)} < 0 \end{cases} \tag{17}$$

and

$$\lambda_i^{(n)} = \begin{cases} \text{given by Equation 16 for } i = n \bmod n_C \\ \lambda_i^{(n-1)}, \qquad \text{for} i \neq n \bmod n_C \end{cases} \tag{18}$$

Calling $e^{\left(\lambda_i^{(n)} - \lambda_i^{(n-1)}\right)} = r_i^{(n)}$, solving Equation 16 is equivalent to calculating the roots of the generalised polynomials

$$P_i(\lambda_i^{(n)}) = \sum_{j;a_j \geq 0} g_{ij} a_j^{(n-1)} r_i^{(n)^{-g_{ij}}} + \sum_{j;a_j < 0} g_{ij} a_j^{(n-1)} r_i^{(n)^{g_{ij}}} - c_i = 0 \tag{19}$$

iteratively for all $\{r_i^{(n)}\}_{i=1,\ldots,n_C}$, given known $p_j^{(n-1)}$ and $\lambda_k^{(n-1)}$.[16]

It is important to understand that the idea of Junius and Oosterhaven of inversely scaling positive and negative matrix elements applies in exactly the same way to elements that are added and subtracted in a constraint, respectively. This is reflected in the sign of the exponents g_{ij} of the r_i. For example, consider the constraint $a_1 + a_2 - a_3 = 2$, with $a_{01} = 3$, $a_{02} = 5$, and $a_{03} = 1$. Scaling a_1 and a_2 with $\frac{1}{2}$, and a_3 with $\frac{1}{2}^{-1} = 2$ yields the desired result $1.5 + 2.5 - 2 = 2$. If an element is both negative and subtracted, it has to be scaled like a positive added element. For example, consider the above constraint with $a_{01} = 3$, $a_{02} = 5$, and $a_{03} = -1$. Scaling all elements a_1, a_2 and a_3 with $^2/_9$ yields the desired result $^6/_9$+$^{10}/_9$ – $(-^2/_9) = 2$.

Note that the g_{ij} are not necessarily unity integers as in GRAS, but $g_{ij} \in \mathbb{R}$. Since, in general, no analytical solution exists, Erlander (1981) and Eriksson (1980) suggest

[16] The problem can be restated so that all exponents are positive, by multiplying Equation 14 with $r_i^{(n)^{\max_{j;p_j \geq 0}\{g_{ij}\}}}$.

Newton's method for an iterative solution.[17] Applying Newton's method to Equation 14 requires calculating the derivative

$$\frac{\partial p_i(\lambda_i^{(n)})}{\partial \lambda_i^{(n)}} = \sum_{j;a_j^{(n-1)}\geq 0} g_{ij} a_j^{(n-1)} e^{-\left(\lambda_i^{(n)}-\lambda_i^{(n-1)}\right)g_{ij}} \prod_k e^{-\lambda_k^{(n-1)}g_{kj}} \left(-g_{ij}\right) + \sum_{j;a_j^{(n-1)}<0} g_{ij} a_j^{(n-1)} e^{\left(\lambda_k^{(n)}-\lambda_i^{(n-1)}\right)g_{ij}} \prod_k e^{\lambda_k^{(n-1)}g_{kj}} g_{ij} \tag{20}$$

and iterating over k

$$\lambda_i^{(n,k)} = \lambda_i^{(n,k-1)} - \frac{p_i(\lambda_i^{(n,k-1)})}{\partial p_i(\lambda_i^{(n,k-1)})/\partial \lambda_i^{(n)}} \tag{21}$$

3.3 Incorporating Reliability and Conflict of External Data

In cases of inconsistent constraints brought about by conflicting external data, the termination condition in Equation 9 may never be met, and GRAS has to be terminated if the distance function between the constraints $\mathbf{c}$ and their realisations $\mathbf{Ga}$ does not improve anymore, that is if for two subsequent iterations $n-1$ and n

$$\|\mathbf{Ga}-\mathbf{c}\|^{(n)} - \|\mathbf{Ga}-\mathbf{c}\|^{(n-1)} < \delta \tag{22}$$

for a sufficiently small δ. Following this termination, we propose a GRAS-type algorithm that modifies the constraints $\mathbf{c}$ as well:

$$\begin{aligned} r^{(n)} &= \frac{c_i^{(n)} + \sqrt{c_i^{(n)2} + 4\sum_j g_{ij}^{+} a_j^{(n-1)} \sum_j g_{ij}^{-} a_j^{(n-1)}}}{2\sum_j g_{ij}^{+} a_j^{(n-1)}}, \\ c_i^{(n)} &= c_i^{(n-1)} - \mathrm{Sgn}\left(c_i^{(n-1)} - \sum_j g_{ij} a_j^{(n-1)}\right) \\ &\times \mathrm{Min}\left(\left|c_i^{(n-1)} - \sum_j g_{ij} a_j^{(n-1)}\right|, \alpha\sigma_i\right) \text{ with } c_i^{(0)} = c_i\,, \\ a_j^{(n)} &= a_j^{(n-1)} \left[r^{(n)}\right]^{\mathrm{Sgn}(g_{ij})}, \text{ and } i = n \bmod n_C, \end{aligned} \tag{23}$$

where $0 \leq \alpha \leq 1$ and the σ_i are the standard deviations of the c_i.[18] We refer to this

[17] Using Newton's method, a root r of a function $f(x)$, in the vicinity of x_0, is approached by first truncating the Taylor expansion around x_0: $f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2} f'(x_0)\ (x - x_0)^2 + \ldots$, to $0 = f(r) \approx f(x_0) + f'(x_0)$ $(r - x_0)$, and then iteratively solving $x_n = x_{n-1} - f(x_{n-1})/f'(x_{n-1})$.

[18] If external data are not normally distributed, the adjustment of constraints $\alpha\sigma_i$ in Equation 23 could be adapted to the characteristics of the alternative distribution. For example, the shape parameter of a Weibull distribution

algorithm as KRAS ("Konfliktfreies RAS"). The essence of this idea is that once GRAS terminates in oscillations without reaching convergence, the original external constraints c_i can clearly not all be satisfied simultaneously, and either some of them or all of them must be erroneous. Just as in constrained optimisation, the c_i must be modified "towards" their realisations $\{\mathbf{Ga}\}_i$ in order to achieve convergence. Since each constraint is known to a higher or lower degree of accuracy. Therefore, an amount $\alpha\sigma_i$ is added or subtracted from each$c_i^{(n-1)}$, depending on the sign $\mathrm{sgn}(c_i^{(n-1)} - \Sigma_j g_{ij} a_j^{(n-1)})$. The constant α can be chosen freely: The higher its value, the more rapid the adjustment process, but also the more inaccurate the adjustment. Note that, in order to prevent overshoot in situations where the realisation $\{\mathbf{Ga}\}_i$ is closer to the c_i than σ_i, the maximum adjustment allowed is $|c_i^{(n-1)} - \Sigma_j g_{ij} a_j^{(n-1)}\,|$. With constraint values modified as in Equation 23, the termination criterion of KRAS is equal to that in Equation 9.

4 APPLICATIONS

In the following, we will apply KRAS to RAS-infeasible balancing problems documented in the literature.

4.1 Cole

The most simple RAS-infeasible problem that we found in the literature was posed by Cole (1992); vectorising Cole's matrix to vector **a**, it can be represented as

$$\mathbf{G} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{c} = \begin{pmatrix} 1 \\ 3 \\ 1 \\ 3 \\ 1 \end{pmatrix} \tag{24}$$

The first four constraints are the column and row sums, respectively, and the fifth constraint fixes a_4. This problem is infeasible for $a_i > 0\ \forall i$, because in requiring $a_4 = 1$, constraint 5 imposes on constraints 2 and 4 that both $a_2 = 2$ and $a_3 = 2$, which conflicts with constraints 1 and 3. Starting with the initial estimate $\mathbf{a}_0 = (1, 1, 1, 1)$, and setting the standard deviation $\boldsymbol{\sigma} = (0.01, 0.01, 0.01, 0.01, 0.01)$,[19] KRAS produces the solution $\mathbf{a} = (0, 1^1/_3, 1^1/_3, 1^1/_3)$ with the realisation $\mathbf{Ga} = (1^1/_3, 2^2/_3, 1^1/_3, 2^2/_3, 1^1/_3)$. The difference between imposed and realised constraint values is $\mathbf{Ga}-\mathbf{c} = (^1/_3, -^1/_3, ^1/_3, -^1/_3, ^1/_3)$, and thus constant across all constraints. This is a direct consequence of the standard deviation $\boldsymbol{\sigma}$ being equal for all constraints.

Setting $\sigma_5 = 0.001$ and $\sigma_5 = 0.1$, respectively, more or less accuracy is put on the fifth constraint (which determines a_4) compared to the column and row sum constraints

could take the place of the standard deviation σ in Equation 23. Further, if conflicting values were distributed uniformly, adjustments could be made in proportion to the uniform ranges. In the case of subjective reliability scores, one could in a similar fashion, that is, by taking normalised scores as step-wise adjustments.

[19] Cole (1992) does not give any information on reliability. All standard deviations were set by the authors.

TABLE 2. KRAS solutions for Cole's (1992) RAS-infeasible problem.*

$\boldsymbol{\sigma}$	0.01	0.01	0.01	0.01	0.001
A	0.00	1.48	1.48	1.05	
Ga	1.48	2.52	1.48	2.52	1.05
Ga − **c**	0.48	−0.48	0.48	−0.48	0.05
$(\mathbf{Ga}-\mathbf{c})\hat{\boldsymbol{\sigma}}^{-1}$	47.57	−47.62	47.61	−47.67	47.66
$\boldsymbol{\sigma}$	0.01	0.01	0.01	0.01	0.1
A	0.00	1.08	1.08	1.83	
Ga	1.08	2.92	1.08	2.92	1.83
Ga − **c**	0.08	−0.08	0.08	−0.08	0.83
$(\mathbf{Ga}-\mathbf{c})\hat{\boldsymbol{\sigma}}^{-1}$	8.27	−8.31	8.32	−8.36	8.34
$\boldsymbol{\sigma}$	0.05	0.04	0.03	0.02	0.01
A	0.00	1.37	1.62	1.13	
Ga	1.62	2.50	1.37	2.75	1.13
Ga − **c**	0.62	−0.50	0.37	−0.25	0.13
$(\mathbf{Ga}-\mathbf{c})\hat{\boldsymbol{\sigma}}^{-1}$	12.46	−12.49	12.50	−12.56	12.56

*Vectors are presented as rows.

(Table 2). Placing more accuracy on constraint 5 yields a solution $a_4 = 1.05$ which is closer to the required value ($\{\mathbf{Ga}-\mathbf{c}\}_5 = 0.05$), while placing less accuracy on constraint 5 yields a solution $a_4 = 1.83$ further away ($\{\mathbf{Ga}-\mathbf{c}\}_5 = 0.83$). The opposite holds for constraint 1 to 4. It is interesting to observe though that the distance $(\mathbf{Ga}-\mathbf{c})\hat{\boldsymbol{\sigma}}^{-1}$ in terms of the number of standard deviations is roughly constant across all constraints in both cases.

Choosing the standard deviation to diminish from constraint 1 through to 5 results in the absolute difference to decrease as well, with constraint 5 satisfied best. Once again, the distance in terms of the number of standard deviations is roughly constant across all constraints. This is a direct consequence of the constraint adjustment mechanism described around Equation 23, which adjusts constraint values $c^{(n)}$ relative to $c^{(n-1)}$ by adding or subtracting a fixed portion a of a one standard deviation. In this case, the final solution is about 12 standard deviations away from the initially prescribed constraint values.

For example, since in the third case the standard deviations of constraints are all different, constraint values are adjusted differently (in absolute terms) throughout the KRAS run. In this case, constraint 1 is assigned the highest standard deviation and its value is therefore altered most (Figure 1).

Once again, as a consequence of the balancing mechanism in Equation 23, the adjustment in terms of the number of standard deviations is constant across all constraints (Figure 2). Throughout the KRAS balancing run, the adjusted constraints move away evenly from the initially prescribed constraints until at about 12 standard deviations, where no more conflict occurs.

In one final numerical experiment on Cole's problem, we increased the range of the σ_i to span 2 orders of magnitude, from 0.001 to 0.1 (Table 3). We observed that the least stringent constraint (number 1) is more closely matched ($9\sigma_1$) than other constraints ($16\sigma_i$).

The reason for this phenomenon is that because of its relatively "lax" standard deviation, constraint 1 starts to become well balanced already from about step 200 in the

FIGURE 1. Evolution of $\Delta\mathbf{c}^{(n)} = \mathbf{c}^{(n)} - \mathbf{c}^{(0)}$ throughout KRAS for Cole's problem.

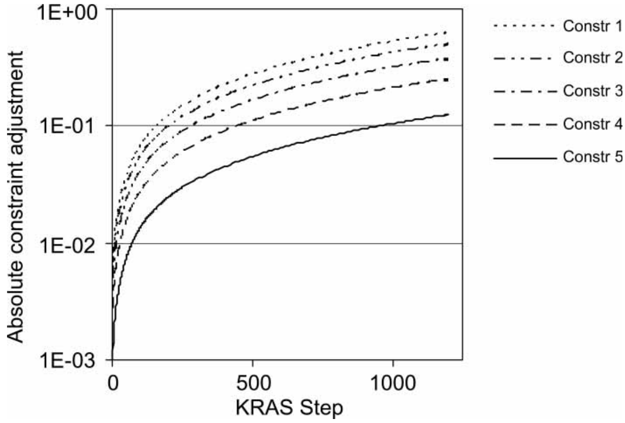


FIGURE 2. Evolution of $\Delta\mathbf{c}^{(n)}\hat{\boldsymbol{\sigma}}^{-1}$ throughout KRAS for Cole's problem.

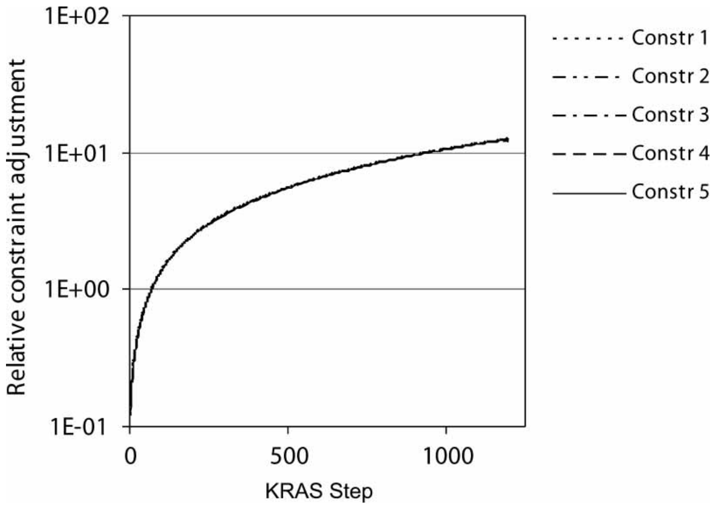
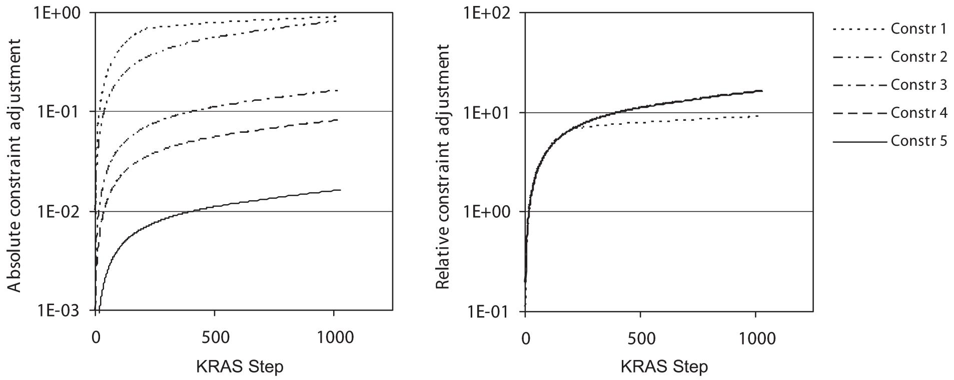


TABLE 3. KRAS solutions for Cole's (1992) RAS-infeasible problem.*

$\boldsymbol{\sigma}$	0.10	0.05	0.01	0.005	0.001
a	0.00	1.16	1.90	1.02	
Ga	1.90	2.18	1.16	2.92	1.02
Ga − c	0.90	−0.82	0.16	−0.08	0.02
$(\mathbf{Ga} - \mathbf{c})\hat{\boldsymbol{\sigma}}^{-1}$	9.02	−16.39	16.39	−16.41	16.40

*Vectors are presented as rows.

FIGURE 3. Evolution of $\Delta\mathbf{c}^{(n)}$ and $\Delta\mathbf{c}^{(n)}\hat{\boldsymbol{\sigma}}^{-1}$ throughout KRAS for Cole's problem.



KRAS run. In other words the realisation $\{\mathbf{Ga}\}_1$ of constraint 1 starts to continuously fall within the interval $[c_1^{(n)} - \alpha\sigma_{1,}\, c_1^{(n)} + \alpha\sigma_1]$. This means that the term $\mathrm{Min}(|c_1^{(n-1)} - \Sigma_{\mathrm{j}} g_{1j} a_j^{(n-1)}|, \alpha\sigma_1)$ in Equation 23 yields $\Delta c_1^{(n)} = c_1^{(n-1)} - \Sigma_{\mathrm{j}} g_{1j} a_j^{(n-1)}$, and the constraint value needs to be adjusted by less than $\alpha\sigma_1$ in order to prevent overshoot (compare Section 0). As a consequence $\Delta\mathbf{c}^{(n)}$ and $\Delta\mathbf{c}^{(n)}\hat{\boldsymbol{\sigma}}^{-1}$ 'flatten out' (Figure 3).

4.2. Möhr et al.

Another RAS-infeasible problem was posed by Möhr et al. (1987). It can be represented as

$$\mathbf{G} = \begin{pmatrix} 1 & 1 & 1 & 1 & & & & & & & & & & & & \\ & & & & 1 & 1 & 1 & 1 & & & & & & & & \\ & & & & & & & & 1 & 1 & 1 & 1 & & & & \\ & & & & & & & & & & & & 1 & 1 & 1 & 1 \\ 1 & & & & 1 & & & & 1 & & & & 1 & & & \\ & 1 & & & & 1 & & & & 1 & & & & 1 & & \\ & & 1 & & & & 1 & & & & 1 & & & & 1 & \\ & & & 1 & & & & 1 & & & & 1 & & & & 1 \end{pmatrix} \text{ and } \mathbf{c} = \begin{pmatrix} 301 \\ 104 \\ 105 \\ 10 \\ 100 \\ 220 \\ 100 \\ 100 \end{pmatrix} \quad (25)$$

The first four constraints are the row sums, the last four constraints are the column sums of the variable matrix

$$\mathbf{A} = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{pmatrix} \quad (26)$$

This problem is RAS-infeasible for the initial estimate

$$\mathbf{A}_0 = \begin{pmatrix} 90 & 0 & 95 & 95 \\ 5 & 101 & 2 & 2 \\ 5 & 101 & 2 & 2 \\ 0 & 18 & 1 & 1 \end{pmatrix} \tag{27}$$

because on the one hand a_1, a_2 and a_4 belong to columns whose sums are required to equal 100 by constraints 1, 2 and 3, hence $a_1 \leq 100$, $a_2 \leq 100$, and $a_4 \leq 100$, but on the other hand $a_1+a_2+a_4 = 301$ is imposed by the first constraint. Strictly speaking, this problem is RAS-infeasible not because of conflicting constraints, but because of ill-set initial estimates. Setting all standard deviations to $\sigma_i = 0.1$, KRAS produces the solution

$$\mathbf{A} = \begin{pmatrix} 100.16 & 0 & 100.26 & 100.26 \\ 0.09 & 104.18 & 0.03 & 0.03 \\ 0.09 & 105.18 & 0.03 & 0.03 \\ 0 & 10.31 & 0.01 & 0.01 \end{pmatrix} \tag{28}$$

with $\mathbf{Ga}$=(300.67, 104.33, 105.33, 10.33, 100.33, 219.67, 100.33, 100.33), $\mathbf{a}$ being the vectorisation of $\mathbf{A}$. All realisations $\{\mathbf{Ga}\}_i$ are 0.33, or 3.3 standard deviations away from the imposed constraints $\mathbf{c}$.

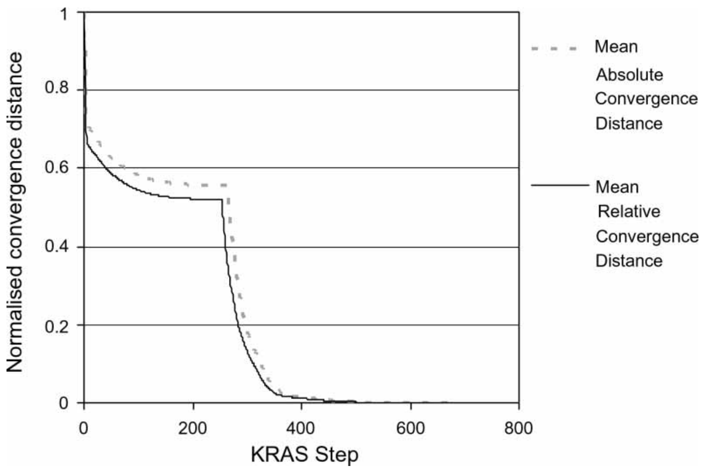
4.3 Updating the Australian Supply-use Tables for 1993–1994

A third example was trialled for a "real-world" scenario that may confront analysts.

Currently the Australian Bureau of Statistics (ABS) provides data within the input–output framework that is balanced between the supply and use tables. This has not always occurred, however, and the 1993–1994 and earlier supply-use tables are not properly balanced across row and column sums. These discrepancies are due to inconsistent source data. For 1993–1994, there is a discrepancy of row totals of up to 11% for 56% of the rows. The final published tables still contain these discrepancies, and the analyst is left with unbalanced tables to use. Further to this issue are the occurrences of confidential cells, which must also be estimated (and is possible with the normal RAS procedure).

The scale of this problem is at the level of 107 products and 107 industries, with seven categories of final demand, and six categories of value added. In this example, the balance is performed across the separate supply and use tables. The data used for both initial estimate and constraints were the 1993–1994 IO tables, published by the Australian Bureau of Statistics (1997). Estimates for the standard deviations of published data were obtained from Australian business operations and industry performance statistics, Australian Bureau of Statistics (1995, 2005) and from financial operating data of Australian manufacturing industries. Additional "balancing constraints" were constructed so as to ensure the balance of row totals, column totals, final and intermediate demand totals, value added totals, and respective row and column totals between the supply and use tables. Standard deviations were set to zero for the balancing constraints so as to ensure that these constraint algorithms were satisfied exactly. The total number of variables is 25,353, and the total number of constraints, $n_C = 26{,}191$.

FIGURE 4. Mean absolute convergence distance and mean relative convergence distance, normalised to their maxima, for the 1993–1994 IO tables.



The balancing run consists of two steps: in a first step a RAS procedure is run until **Ga** stops converging towards $\mathbf{c}^{(n)} = \mathbf{c}^{(0)}$, which occurs when adjustment cannot continue with the given conflicting constraint values (here after 264 iterations); in a second step a KRAS procedure allows the constraints to change according to their assigned deviations. The initial RAS step is implemented in order to estimate confidential cells and other imbalances in the table before KRAS is applied. Convergence is assumed when the maximum deviation of any $(\mathbf{Ga}^n)_i$ from$c_i^{(n)}$ is less than \$1 (here after 681 iterations). Figure 4 shows the evolution of the mean absolute convergence distance between **Ga** and $\mathbf{c}^{(n)} : \Sigma_i|\mathbf{Ga}^n - \mathbf{c}^{(n)}|_i/n_C$, and of the mean relative convergence distance: $\Sigma_i|\mathbf{Ga} - \mathbf{c}^{(n)}|_i/([\sigma_i + \delta]n_C)$ throughout the RAS and KRAS steps (up to iteration number 1500)[20]. Values have been normalised to their respective maxima.

Figure 4 shows the initial step during normal RAS convergence, with preliminary rapid gains diminishing at approximately half of the required convergence distance to the final solution. This "premature" convergence is entirely due to constraint conflict. At the 265th iteration, KRAS allows further convergence, with rapid initial gains (corresponding to the adjustment of the most problematic constraints) again slowing as true convergence is reached, and finer adjustments are required.

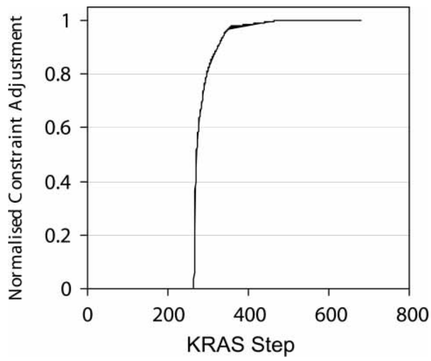
In Figure 5, the mean deviation of $c^{(n)}$ from $c^{(0)}$,$\Sigma_i|\mathbf{c}^{(n)} - \mathbf{c}^{(0)}|_i/n_C$, is plotted, again normalised to the maximum deviation (we call this the 'mean constraint adjustment').

The initial 264 iterations follow the normal RAS procedure, and hence do not require constraint adjustment. The ensuing iterations, however, show large initial changes, corresponding to the changing of the most inaccurate constraints. The speed at which this adjustment takes place (and hence the slope of the graph) can be controlled through changing the precision of the adjustment parameter, α.[21]

[20] The small δ value has been incorporated to handle the case of $\sigma_i = 0$.

[21] The run time for this example was in the order of several seconds for complete covergence, using the intel fortran90 compiler v.7.1 on a RHEL-WS24 linux Kernel with a Xenon 32 bit and 2×3.1 GHz CPU and 4Gb RAM.

FIGURE 5. Mean constraint adjustment by iteration, normalised to the maximum adjustment, for the 1993–1994 IO tables.



4.4 An Australian Multi-regional Input–output System

In a fourth case study, we have used KRAS to balance a detailed multi-regional input–output (MRIO) framework of Australia. This framework covers 344 industrial sectors across the eight Australian States and Territories.[22] It makes use of a large and diverse amount of the most recent survey data available (for details see Gallego and Lenzen, 2009). To our knowledge, this is the first such detailed framework estimated in Australia. Furthermore, this MRIO system has been complemented with physical data on more than a thousand social and environmental parameters, such as employment, water use or greenhouse gas emissions.[23] This generalisation allows tracing social and environmental impacts along interregional supply chains. The trade specification in this model has a strong influence on evening out regional differences of environmental intensities of commodities bought by consumers. For example, due to Australia's variable climate, land and water use intensities in agriculture vary significantly across regions. These intensities become "mixed" in a final consumer's commodity basket due to interregional trade. The extent of this dilution, and hence the magnitude of land and water intensities of regionally purchased commodities, depends critically on the specification of the trade model.

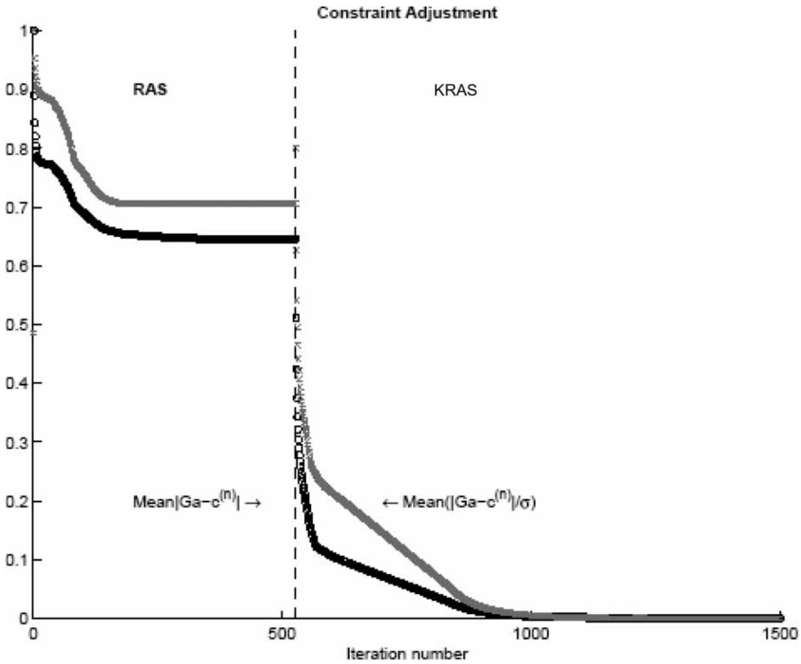
A preliminary estimate of the Australian 1998–1999 economic MRIO system, which contains 8,792,675 variables, has been balanced to satisfy $n_C = 462{,}909$ constraints using KRAS. These constraints take different shapes and sizes ranging from 1 coefficient constraints to constraints containing more than 8 million coefficients. The coefficients, in turn, include both negative as well as non-unity real values. Furthermore, many of the constraints come from survey data such as the national input–output tables, state accounts, and other commodity statistics, and contain errors which make them conflicting. Therefore they have been assigned standard deviations $\{\sigma_i\}$ according to their origin.[24] A smaller

[22] New South Wales, Victoria, Queensland, South Australia, Western Australia, Tasmania, Northern Territory, Australian Capital Territory.

[23] See http://www.isa.org.usyd.edu.au/research/ISA_TBL_Indicators.pdf.

[24] It has been assumed that these $\{\sigma i\}$ represent the standard deviations of normally distributed random errors in the survey data. Possible systematic errors in the data sources have not been captured in this model.

FIGURE 6. Evolution of the mean absolute convergence distance and mean relative convergence distance, normalised to their maxima, throughout RAS and the first two thirds of KRAS during the balancing of the Australian MRIO system.



fraction of the constraints, such as the internal accounting constraints are to be satisfied exactly and are assigned deviations $\sigma_i = 0$.[25]

As per the Australian 1993–1994 tables, the balancing run consists of two steps: the initial RAS procedure until **Ga** stops converging towards $\mathbf{c}^{(n)} = \mathbf{c}^{(0)}$ (here after 526 iterations); and the subsequent KRAS procedure. Convergence is assumed when the maximum deviation of any $(\mathbf{Ga}^n)_i$ from$c_i^{(n)}$ is less than \$1000 (here after 3328 iterations). Figure 6 shows the evolution of the mean absolute convergence distance between **Ga** and $\mathbf{c}^{(n)}$ and of the mean relative convergence distance throughout the RAS and KRAS steps (up to iteration number 1500), as in Figure 4. Values have again been normalised to their respective maxima.

Both balancing steps are characterised by a rapid initial adjusting phase (first third of the full step) followed by a slow adjustment towards the final values which extends over the remaining iterations. The evolution of the normalised average change in constraint values during the first 1500 KRAS iterations is shown in Figure 7 (which has a similar setup as Figure 5).[26]

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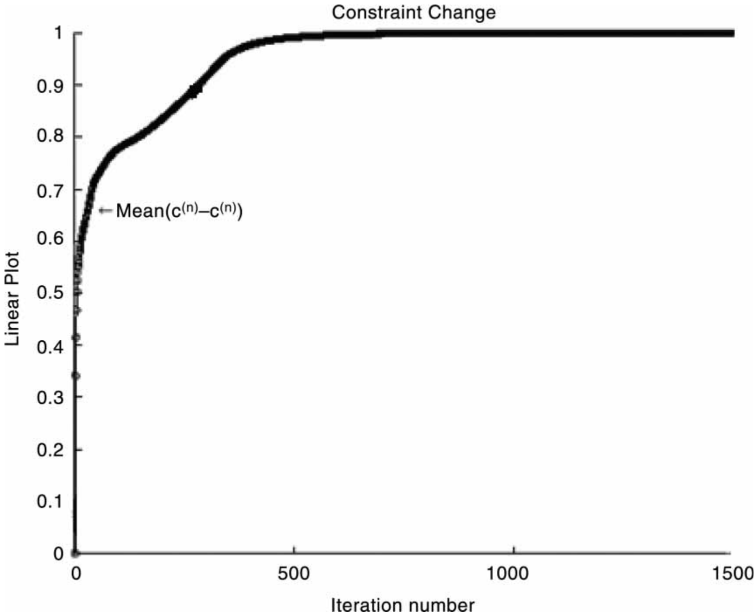
[25] A small δ value was applied in cases of $\sigma i = 0$.

[26] The initial RAS iterations took up to 5 min of run time each, due to the time spent on the Newton algorithm looking for a solution of Equation 16 when far from the initial estimate. However, this time shortened as the run advanced and the average run time for a KRAS iteration was about 5 sec.

FIGURE 7. Evolution of the mean constraint adjustment by iteration, normalised to the maximum adjustment, during the balancing of the Australian MRIO system.



5 CONCLUSIONS

We have developed a RAS variant, generalised iterative scaling method (KRAS) that is able to balance and reconcile input–output tables and SAMs under conflicting external information and inconsistent constraints. In addition, KRAS fulfils all requirements of earlier RAS variants such as GRAS, and of constrained optimisation techniques:

a) constraints on arbitrarily sized and shaped subsets of matrix elements;
b) consideration of the reliability of the initial estimate and the external constraints;
c) ability to handle negative values and to preserve the sign of matrix elements.

Applying KRAS in four case studies, we find that, as with constrained optimisation, KRAS is able to find a compromise solution between inconsistent constraints; this feature does not exist in conventional RAS variants such as GRAS.

KRAS can constitute a major advance for the practice of balancing input–output tables and Social Accounting Matrices, in that it removes the necessity of manually tracing inconsistencies in external information. In contrast to constrained optimisation, this quality does not come at the expense of substantial programming requirements, and long run times.

While KRAS appears to be able to solve these problems, it should not be forgotten that collecting data requires skilled manual input. As Barker et al. emphasise, automated balancing of national accounts “is not a replacement for knowledge of the data and its sources but an enhancement of it, allowing us to produce fully balanced accounts with the adjustments reflecting the quality of the data (Barker et al., 1984).”

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